SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2022 Trial HSC Examination

Form VI Mathematics Extension 2

Wednesday 10th August 2022
8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 100

Section I (10 marks) Questions 1 – 10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11 – 16
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.
- Place everything inside this question booklet.

Checklist
- Reference sheet
- Multiple-choice answer sheet
- 6 booklets per boy
- Candidature: 78 pupils

Writer: WJM

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. What is the value of $(-1+i)^2$?

   (A) -2

   (B) 2

   (C) $-2i$

   (D) $2i$

2. Consider the statement:

   If Bob eats an apple every day, then he does not have to visit a doctor.

   What is the **converse** of this statement?

   (A) If Bob does not eat an apple every day, then he has to visit a doctor.

   (B) If Bob eats an apple every day, then he has to visit a doctor.

   (C) If Bob has to visit a doctor, then he does not eat an apple every day.

   (D) If Bob does not have to visit a doctor, then he eats an apple every day.

3. The two lines $\underset{\sim}{r_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\underset{\sim}{r_2} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ intersect at the point $(2,\, 5,\, -1)$.

   What is the approximate size of the angle between $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$?

   (A) 50°

   (B) 82°

   (C) 97°

   (D) 140°

4. Which of the following integrals is equivalent to $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\, dx$?

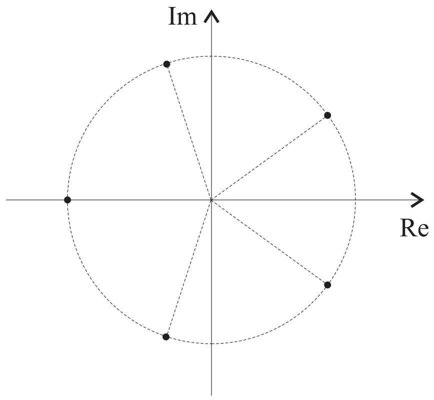
   (A) $\displaystyle\int \sqrt{1-x^2}\, dx$

   (B) $\displaystyle\int \frac{1+x}{\sqrt{1-x^2}}\, dx$

   (C) $\displaystyle\int \frac{1+x}{1-x}\, dx$

   (D) $\displaystyle\int \left(1 + \frac{2x}{\sqrt{1-x}}\right) dx$

5.



The diagram above shows the solutions to $z^5 + a = 0$. Which of the following could be the value of a?

(A) 2

(B) -3

(C) $4i$

(D) $-5i$

6. Euler's *sum of powers* conjecture states that for $a_1, a_2, \ldots, a_n, b \in \mathbf{Z}$:

If $\exists\, k \in \mathbf{Z}^+$ such that ${a_1}^k + {a_2}^k + \ldots + {a_n}^k = b^k$, then $n \geq k$.

Given that each of the following statements is true, which statement is a counterexample to this conjecture?

(A) $27^2 + 72^2 + 96^2 = 123^2$

(B) $133^4 + 134^4 - 59^4 = 158^4$

(C) $27^5 + 84^5 + 110^5 + 133^5 = 144^5$

(D) $955^4 + 1770^4 + 2634^4 + 5400^4 = 5491^4$

7. The velocity of an object in metres per second is defined by $\dot{x} = 6\cos \pi t$. What distance does the object travel from $t = 0$ to $t = 1$?

(A) $\dfrac{6}{\pi}$ metres

(B) $\dfrac{12}{\pi}$ metres

(C) 6 metres

(D) 12 metres

8. The complex number z is defined such that $|z| = |z - 1 - i|$. Which of the following describes the set of values $\mathrm{Arg}(z)$ can take?

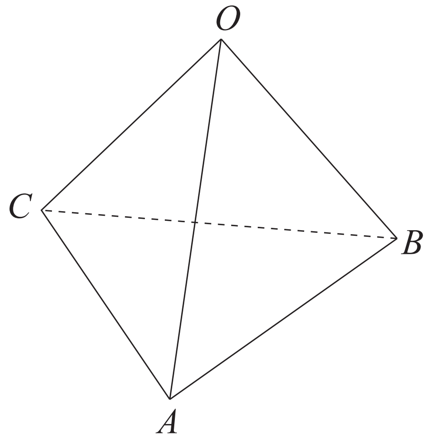
(A) $-\pi < \mathrm{Arg}(z) \leq \pi$

(B) $\mathrm{Arg}(z) = -\frac{3\pi}{4}$ or $\mathrm{Arg}(z) = \frac{\pi}{4}$

(C) $-\frac{\pi}{2} < \mathrm{Arg}(z) < \frac{\pi}{2}$

(D) $-\frac{\pi}{4} < \mathrm{Arg}(z) < \frac{3\pi}{4}$

9.



The diagram above shows the tetrahedron $OABC$. That is, $OABC$ is a triangular pyramid where each face is an equilateral triangle. Each edge of the pyramid has length 1 unit.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, and $\overrightarrow{OC} = \underset{\sim}{c}$.

What is the value of $\left|\text{proj}_{\underset{\sim}{a}}\left(\text{proj}_{\underset{\sim}{c}}(\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a})\right)\right|$?

(A) $\dfrac{1}{8}$

(B) $\dfrac{1}{4}$

(C) 1

(D) 4

10. Consider the following proof:

For $a,\ b,\ c \in \mathbf{Z}^+$, $\exists\ p,\ q,\ r \in \mathbf{Z}^+$ such that $a = 3p+1$, $b = 3q+1$, $c = 3r+1$.

$$\begin{aligned} abc &= (3p+1)(3q+1)(3r+1) \\ &= 27pqr + 9pq + 9pr + 9qr + 3p + 3q + 3r + 1 \\ &= 3(9pqr + 3pq + 3pr + 3qr + p + q + r) + 1 \\ &= 3k+1, \quad k \in \mathbf{Z}^+ \end{aligned}$$

Which of the following can be deduced from the proof above?

(A) If the product of three integers is one more than a multiple of 3, then the three integers must all be one more than a multiple of 3.

(B) If the product of three integers is one more than a multiple of 3, then at least one of the three integers must be one more than a multiple of 3.

(C) If the product of three integers is two more than a multiple of 3, then none of the three integers can be one more than a multiple of 3.

(D) If the product of three integers is two more than a multiple of 3, then at least one of the three integers must not be one more than a multiple of 3.

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (16 marks) Start a new answer booklet. **Marks**

(a) Given the two complex numbers $z = 1 + \sqrt{3}\,i$ and $w = 3e^{i\frac{2\pi}{3}}$,

(i) Express z in the form $re^{i\theta}$. $\boxed{2}$

(ii) Find zw, fully simplifying your answer. $\boxed{2}$

(b) Find:

(i) $\displaystyle\int \frac{\sec^2 x}{1 + 2\tan x}\,dx$ $\boxed{1}$

(ii) $\displaystyle\int xe^x\,dx$ $\boxed{2}$

(iii) $\displaystyle\int \cos^3 x\,dx$ $\boxed{2}$

(c) Solve $z^2 - 2iz - 5 = 0$. $\boxed{2}$

(d) Consider the statement: $\forall x \in \mathbf{R}, \exists\, y \in \mathbf{R}$ such that $y^2 = x^3$. $\boxed{1}$

Determine if this statement is true or false. Justify your answer.

(e) Given the complex numbers $z = 1 + 5i$ and $w = 3 - 2i$, find the real numbers a and b such that $\dfrac{z}{\overline{w}} = a + ib$. $\boxed{2}$

(f) Consider the line $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The point $A(x,\, y)$ lies on the line, whereas the point $B(-1,\, 3)$ does not. $\boxed{2}$

If O is the origin, find the coordinates of A if $\overrightarrow{OA} \perp \overrightarrow{OB}$.

QUESTION TWELVE (15 marks) Start a new answer booklet. **Marks**

(a) For two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, it is known that $|\underset{\sim}{a}| = 5$, $|\underset{\sim}{b}| = 3$, and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 9$.

(i) Show that $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$. $\boxed{1}$

(ii) Find $|\underset{\sim}{a} - \underset{\sim}{b}|$. $\boxed{2}$

(iii) If $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, find the area of $\triangle OAB$. $\boxed{1}$

(b) (i) Prove that if x and y are positive real numbers, then $\dfrac{x+y}{2} \geq \sqrt{xy}$. $\boxed{1}$

(ii) The numbers $a_1, a_2, a_3, \ldots, a_n$ are positive and real, and $a_1 a_2 a_3 \ldots a_n = 1$. $\boxed{2}$

Show that $(1 + a_1)(1 + a_2)(1 + a_3) \ldots (1 + a_n) \geq 2^n$.

(c) Use the substitution $t = \tan \frac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \sin\theta}$. $\boxed{4}$

(d) Let $I_n = \displaystyle\int_0^1 \frac{x^n}{x+1}\, dx$ for $n \geq 0$.

(i) Use algebraic manipulation to show that $I_n = \dfrac{1}{n} - I_{n-1}$ for $n \geq 1$. $\boxed{2}$

(ii) Hence find I_2. $\boxed{2}$

QUESTION THIRTEEN (14 marks) Start a new answer booklet. **Marks**

(a) The polynomial $P(z) = z^4 - 10z^3 + cz^2 + dz + 169$ has two zeroes $\alpha = a + ib$ and $\beta = b + ia$, where a, b, c, $d \in \mathbf{R}$ and $a \neq b$.

(i) Express the other two zeroes of $P(z)$ in terms of a and b, justifying your answer. $\boxed{1}$

(ii) Find the values of a and b, given that they are both integers. $\boxed{2}$

(b) A particle moves in a straight line with initial displacement $x = 0$. The velocity of the particle is given by $v = 2e^{-\frac{x}{2}}(x+1)^2$, where velocity is in metres per second.

(i) Show that the acceleration of the particle is given by $a = 2e^{-x}(x+1)^3(3-x)$. $\boxed{2}$

(ii) Hence find the displacement for which the maximum velocity of the particle will occur, justifying your answer. $\boxed{2}$

(c) (i) Use De Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. $\boxed{3}$

(ii) Consider the equation $x^4 - 4x^2 + 1 = 0$. Use the substitution $x = 2\cos\theta$ to show that the roots are $2\cos\frac{\pi}{12}$, $2\cos\frac{5\pi}{12}$, $2\cos\frac{7\pi}{12}$, and $2\cos\frac{11\pi}{12}$. $\boxed{2}$

(iii) Hence show that $\cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + \cos^2\frac{7\pi}{12} + \cos^2\frac{11\pi}{12} = 2$. $\boxed{2}$

QUESTION FOURTEEN (14 marks) Start a new answer booklet. **Marks**

(a) Three points $A(-2,\, 1,\, 7)$, $B(-1,\, 3,\, 5)$, and $C(-4,\, 7,\, 3)$ lie in three-dimensional space. The points A and B lie on the line l.

(i) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors. **1**

(ii) Find the shortest distance from C to the line l. **3**

(b) For constants p, A, and B, $\dfrac{1}{p^2 - x^2} = \dfrac{A}{p+x} + \dfrac{B}{p-x}$.

(i) Find expressions for A and B in term of p. **2**

(ii) Hence or otherwise, show that for some constant C, **1**

$$\int \frac{1}{p^2 - x^2}\, dx = \frac{1}{2p} \ln \left| \frac{p+x}{p-x} \right| + C\,.$$

(c) A physicist drops an object with mass m kg from the top of a cliff. As the object falls, it experiences a force due to gravity of magnitude $10m$ Newtons and a resistive force of magnitude mkv^2 Newtons, where v is the velocity of the object in metres per second and k is a positive constant.

The vertical displacement of the object y metres from where it is dropped satisfies

$$m\ddot{y} = 10m - mkv^2\,,$$

where the downwards direction is taken as positive.

(i) Let V be the terminal velocity of the object. Find an expression for V in terms of k, and hence show that $\dfrac{dv}{dt} = k(V^2 - v^2)$. **2**

(ii) Use part (b)(ii) to show that **3**

$$v = V \times \frac{e^{2Vkt} - 1}{e^{2Vkt} + 1}\,.$$

(iii) The physicist finds that it takes the object 1 second to reach half of its terminal velocity. Find the value of k. Give your answer correct to two decimal places. **2**

QUESTION FIFTEEN (16 marks) Start a new answer booklet. **Marks**

(a) Sketch the region on the Argand diagram where $\mathrm{Im}\left(\dfrac{z+\overline{z}}{z-\overline{z}}\right) \geq 1$. $\boxed{3}$

(b) (i) Write down the roots of the equation $z^7 = 1$. $\boxed{2}$

(ii) Hence show that $\boxed{2}$

$$z^6+z^5+z^4+z^3+z^2+z+1 = \left(z^2 - 2z\cos\tfrac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\tfrac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\tfrac{6\pi}{7} + 1\right).$$

(iii) Deduce that $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$. $\boxed{2}$

(c) Use mathematical induction to show that $\left(2 - \dfrac{1}{n}\right)^n > n$ for all integers $n \geq 2$. $\boxed{3}$

(d) (i) The function $f(x)$ is continuous and differentiable for all real x. $\boxed{2}$

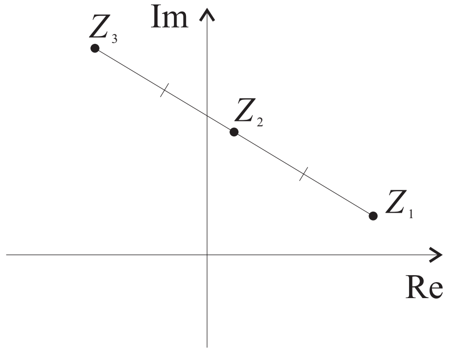
Use the substitution $u = a + b - x$ to show that

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx = \frac{b-a}{2}.$$

(ii) Hence use the substitution $u = x^2$ to evaluate $\displaystyle\int_1^3 \frac{x^2}{\sqrt{10-x^2}+x}\,dx$. $\boxed{2}$

QUESTION SIXTEEN (15 marks) Start a new answer booklet. **Marks**

(a)



The diagram above shows the collinear points Z_1, Z_2, and Z_3, which represent the complex numbers z_1, z_2, and z_3 respectively. The midpoint of the interval between Z_1 and Z_3 is Z_2, and it is also known that $z_3 = z_1 z_2$.

(i) Show that $z_2 = \dfrac{z_1}{2 - z_1}$. **1**

(ii) Consider the scenario where z_2 is purely imaginary.

(α) Show that Z_1 will always lie on a particular circle. Find the centre and radius of this circle. **2**

(β) Show that the line through Z_1, Z_2 and Z_3 is a tangent to the circle mentioned in part (ii)(α). **2**

(b) (i) By considering the dot product of two vectors, show that for any $a_k,\, b_k \in \mathbf{R}$, **2**

$$\left({a_1}^2 + {a_2}^2 + {a_3}^2\right)\left({b_1}^2 + {b_2}^2 + {b_3}^2\right) \geq (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 .$$

(ii) Given that x, y, and z are non-negative and $x + y + z = 3$, find the minimum possible value of $x^2 + 4y^2 + z^2$. **2**

(c) A function $f(x)$ is continuous and positive in its natural domain, which includes the interval $[0, 3]$.

It is known that $f(3) = 5$, and the function also has the following properties:

- $\displaystyle\int_0^3 \big(f(x)\big)^2\, dx = 81$
- $\displaystyle\int_0^3 \big(f'(x)f(x)\big)^2\, dx = 1$

(i) Show that $\displaystyle\int_0^3 x f'(x) f(x)\, dx = -3$ **3**

(ii) By considering $\big(f'(x)f(x) + kx\big)^2$, or otherwise, find $f(0)$. **3**

END OF PAPER

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- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Mathematics Ext. 2 Trial - 2022 Solutions

Multiple Choice

(1) $(-1+i)^2 = 1-2i+i^2$

$= -2i$ (C)

(2) (D)

(3) $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}|\times|\underset{\sim}{b}|\times\cos\theta$

$\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\times|\underset{\sim}{b}|}$

$|\underset{\sim}{a}| = \sqrt{1^2+2^2+(-1)^2}$

$= \sqrt{6}$

$|\underset{\sim}{b}| = \sqrt{0^2+1^2+3^2}$

$= \sqrt{10}$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \begin{bmatrix}1\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\3\end{bmatrix}$

$= 0+2-3$

$= -1$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{-1}{\sqrt{6}\times\sqrt{10}}\right)$

$\doteqdot 97^\circ$ (C)

(4) $\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1+x}{\sqrt{1-x}\sqrt{1+x}}$

$= \dfrac{1+x}{\sqrt{1-x^2}}$ (B)

(5) One solution is negative and real.

$\therefore$ z^5 is negative and real

$\therefore$ a is positive and real (A)

(6) In $27^5+84^5+110^5+133^5 = 144^5$, $n=4$ and $k=5$

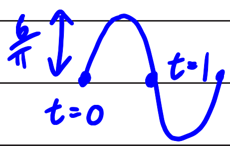
so $n<k$

$\therefore$ (C)

(7) $x = \frac{6}{\pi}\sin\pi t$ (letting $x=0$ be centre of motion)

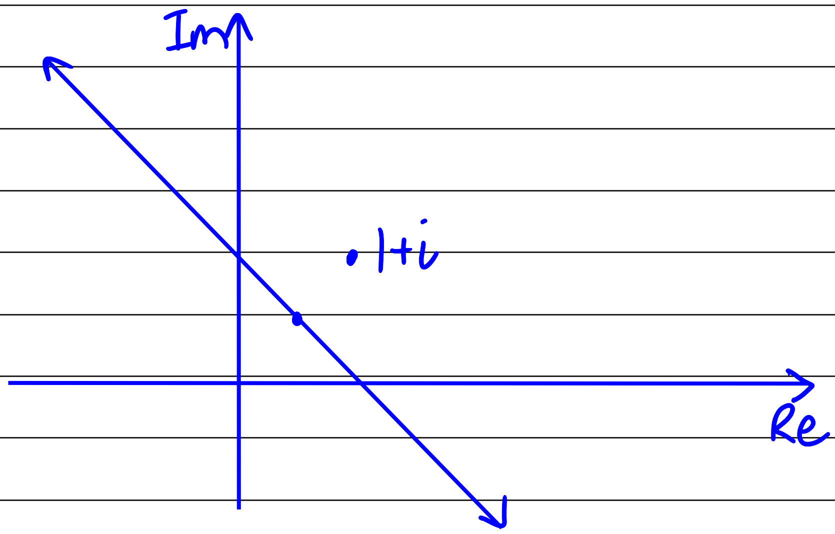
Period $= \dfrac{2\pi}{\pi}$

$= 2$



$2\times\frac{6}{\pi} = \dfrac{12}{\pi}$ $\Rightarrow$ (B)

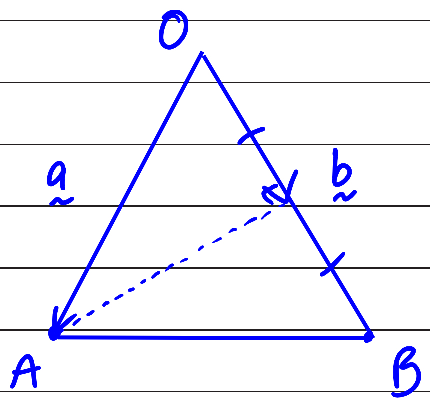
(8)



For any z on this line, $-\frac{\pi}{4} < \text{Arg}(z) < \frac{3\pi}{4}$

$\therefore$ D

(9)



$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$

So $|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}| = \frac{1}{2}$

Each projection halves the length of the previous one.

$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ A

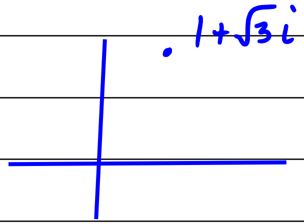
(10) The proof states:

If a, b, c are 1 more than a multiple of 3, then abc is 1 more than a multiple of 3.

Contrapositive: If abc is <u>not</u> 1 more than a multiple of 3, then <u>not all of</u> a, b, c are 1 more than a multiple of 3. D fits this structure

Question 11

(a) (i) $z = 1 + \sqrt{3}\,i$



$|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\text{Arg}(z) = \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$

$\therefore \quad z = 2e^{i\frac{\pi}{3}}$

(ii) $zw = 2e^{i\pi/3} \times 3e^{i2\pi/3}$

$= 6e^{i\pi}$

$= -6$

(b) (i) $\frac{1}{2}\int \frac{2\sec^2 x}{1 + 2\tan x}\,dx = \frac{1}{2}\ln|1 + 2\tan x| + c$

(ii) $\int xe^x\,dx = \int x \times \frac{d}{dx}(e^x)\,dx$ (applying IBP)

$= xe^x - \int e^x\,dx$

$= xe^x - e^x + c$

(iii) $\int \cos^3 x\,dx = \int \cos x(1 - \sin^2 x)\,dx$ let $u = \sin x$

$du = \cos x\,dx$

$= \int (1 - u^2)\,du$

$= u - \frac{u^3}{3} + c$

$= \sin x - \frac{1}{3}\sin^3 x + c$

(c) $z^2 - 2iz - 5 = 0$

$(z - i)^2 = 5 + i^2$

$(z - i)^2 = 4$ (finding Δ if using quadratic formula)

$z - i = \pm 2$

$z = \pm 2 + i$

So $z = 2 + i$ or $z = -2 + i$

(d) False. If $x < 0$, then $y^2 < 0$ so y can't be real. ✓

(e) $\frac{z}{\bar{w}} = \frac{1+5i}{\overline{3-2i}}$

$= \frac{1+5i}{3+2i} \times \frac{3-2i}{3-2i}$ ✓

$= \frac{3+10+i(-2+15)}{9+4}$

$= \frac{13+13i}{13}$

$= 1+i$ ✓ $\therefore a=1, \ b=1$

(f) $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 4+\lambda \\ -2+2\lambda \end{bmatrix}$

$\therefore \vec{OA} = \begin{bmatrix} 4+\lambda \\ -2+2\lambda \end{bmatrix}$

$\vec{OA} \cdot \vec{OB} = \begin{bmatrix} 4+\lambda \\ -2+2\lambda \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ ✓ (correctly using dot product)

$= -4-\lambda -6+6\lambda$

$= -10+5\lambda$

$\therefore \vec{OA} \cdot \vec{OB} = 0$ when $\lambda = 2$

$\therefore \vec{OA} = \begin{bmatrix} 4+2 \\ -2+2\times 2 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\therefore$ A is the point $(6, 2)$. ✓

Question 12

(a) (i)
$$\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$$
$$= \underset{\sim}{b} \cdot \underset{\sim}{a} - |\underset{\sim}{b}|^2$$
$$= 9 - 3^2$$
$$= 0 \quad \checkmark$$

(ii)
$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$$
$$= \underset{\sim}{a} \cdot \underset{\sim}{a} - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b} \quad \checkmark$$
$$= |\underset{\sim}{a}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2$$
$$= 5^2 - 2 \times 9 + 9$$
$$= 16$$
$$\therefore |\underset{\sim}{a} - \underset{\sim}{b}| = \sqrt{16}$$
$$= 4 \quad \checkmark$$

(iii) From (i), $\overrightarrow{OB} \perp \overrightarrow{BA}$
$$\therefore \text{Area} = \frac{1}{2} \times |\overrightarrow{OB}| \times |\overrightarrow{BA}|$$
$$= \frac{1}{2} \times |\underset{\sim}{b}| \times |\underset{\sim}{a} - \underset{\sim}{b}|$$
$$= \frac{1}{2} \times 3 \times 4$$
$$= 6 \text{ units}^2 \quad \checkmark$$

(b) (i)
$$(\sqrt{x} - \sqrt{y})^2 \geq 0$$
$$\therefore x + y - 2\sqrt{xy} \geq 0$$
$$x + y \geq 2\sqrt{xy}$$
$$\frac{x+y}{2} \geq \sqrt{xy} \quad \checkmark$$

Note: Squaring both sides of an inequality was penalised if proper justification wasn't given.

(ii) Using (i):
$$1 + a_1 \geq 2\sqrt{a_1}$$
$$1 + a_2 \geq 2\sqrt{a_2}$$
$$1 + a_3 \geq 2\sqrt{a_3}$$
$$\vdots$$
$$1 + a_n \geq 2\sqrt{a_n}$$

✓ (any correct application of AM-GM)

Multiplying:
$$(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n) \geq 2\sqrt{a_1} \times 2\sqrt{a_2} \times 2\sqrt{a_3} \times \ldots \times 2\sqrt{a_n}$$
$$\geq 2^n \sqrt{a_1 a_2 a_3 \ldots a_n}$$
$$\geq 2^n \text{ as } a_1 a_2 a_3 \ldots a_n = 1$$
✓

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta}$

Let $t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$= \frac{1}{2}(1+t^2)$

$\therefore d\theta = \frac{2\,dt}{1+t^2}$

$\theta = 0, \quad t = 0$

$\theta = \frac{\pi}{2}, \quad t = 1$

$$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}$$

$$= \int_0^1 \frac{2\,dt}{1+t^2+2t}$$

$$= 2\int_0^1 \frac{dt}{(1+t)^2}$$

$$= 2\left[-(1+t)^{-1}\right]_0^1$$

$$= 2\left[-\frac{1}{1+t}\right]_0^1$$

$$= 2\left(-\frac{1}{1+1} + \frac{1}{1}\right)$$

$$= 2 \times \frac{1}{2}$$

$$= 1$$

(d) $I_n = \displaystyle\int_0^1 \frac{x^n}{x+1}\,dx, \quad n \geq 0$

(i) $$I_n = \int_0^1 \frac{(x+1)x^{n-1} - x^{n-1}}{x+1}\,dx$$

$$= \int_0^1 \left(x^{n-1} - \frac{x^{n-1}}{x+1}\right)dx$$

$$= \left[\frac{x^n}{n}\right]_0^1 - I_{n-1}$$

$$= \frac{1}{n} - I_{n-1}$$

(ii) $I_0 = \int_0^1 \frac{dx}{x+1}$

$= \left[\ln(x+1)\right]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$ ✓

$\therefore I_1 = \frac{1}{1} - \ln 2$

$= 1 - \ln 2$

$I_2 = \frac{1}{2} - (1 - \ln 2)$

$= \ln 2 - \frac{1}{2}$ ✓

Question 13

(a) (i) The coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs.

$\therefore$ the other zeroes are $a-ib$ and $b-ia$. ✓

(ii) Sum of zeroes:

$$(a+ib)+(a-ib)+(b+ia)+(b-ia)=10$$
$$2a+2b=10$$
$$a+b=5 \quad (1)$$

✓ (either eqn. involving a and b)

Product of zeroes:

$$(a+ib)(a-ib)(b+ia)(b-ia)=169$$
$$(a^2+b^2)(b^2+a^2)=169$$
$$(a^2+b^2)^2=169$$
$$a^2+b^2=13, \quad \text{as } a,b\in\mathbb{R}$$

$(1)^2$:
$$a^2+2ab+b^2=25$$
$$2ab+13=25$$
$$ab=6$$ ✓

By inspection, as a & b are integers, $a=2$, $b=3$ (or $b=2$, $a=3$).

(b) (i)
$$v=2e^{-\frac{x}{2}}(x+1)^2$$
$$v^2=4e^{-x}(x+1)^4$$
$$\tfrac{1}{2}v^2=2e^{-x}(x+1)^4$$ ✓ (method)
$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right)=2\left(-e^{-x}(x+1)^4+e^{-x}\times 4(x+1)^3\right)$$

$$\therefore \ddot{x}=2e^{-x}(x+1)^3\left[4-(x+1)\right]$$
$$=2e^{-x}(x+1)^3(3-x)$$ ✓

(ii) Note that $v>0\ \forall x$. Initially, $v=2\times e^0(0+1)^2=2\text{ms}^{-1}$, so particle always has positive displacement after $t=0$. $\ddot{x}=0$ when $x=3$. ✓ Also, $\ddot{x}>0$ for $0<x<3$ and $\ddot{x}<0$ for $x>3$. ✓ $\therefore$ Max. velocity occurs when $x=3$.

$\boxed{\text{Note: } \ddot{x}=0 \text{ when } x=3 \text{ is not sufficient justification, as it ignores the possibility of the object speeding up for } x>3 \text{ (i.e. } v \text{ having an inflection at } x=3\text{).}}$

(c) (i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ by De Moivre's theorem

$$(\cos\theta + i\sin\theta)^4 = \binom{4}{0}\cos^4\theta + \binom{4}{1}\cos^3\theta(i\sin\theta) + \binom{4}{2}\cos^2\theta(i\sin\theta)^2 + \binom{4}{3}\cos\theta(i\sin\theta)^3 + \binom{4}{4}(i\sin\theta)^4$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

Equating real: $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

(ii) $x^4 - 4x^2 + 1 = 0$

Let $x = 2\cos\theta$:

$$(2\cos\theta)^4 - 4\times(2\cos\theta)^2 + 1 = 0$$

$$16\cos^4\theta - 16\cos^2\theta + 1 = 0$$

$$8\cos^4\theta - 8\cos^2\theta + \tfrac{1}{2} = 0$$

$$8\cos^4\theta - 8\cos^2\theta + 1 = \tfrac{1}{2}$$

$$\cos 4\theta = \tfrac{1}{2}$$

So the roots of the equation are $x = 2\cos\theta$, where $\cos 4\theta = \frac{1}{2}$

$\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}$ (taking further values gives duplicate results for $\cos\theta$)

$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

$\therefore$ roots are $2\cos\frac{\pi}{12}, 2\cos\frac{5\pi}{12}, 2\cos\frac{7\pi}{12}, 2\cos\frac{11\pi}{12}$

(iii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$\therefore 4\cos^2\frac{\pi}{12} + 4\cos^2\frac{5\pi}{12} + 4\cos^2\frac{7\pi}{12} + 4\cos^2\frac{11\pi}{12} = 0^2 - 2\times -4$

$= 8$

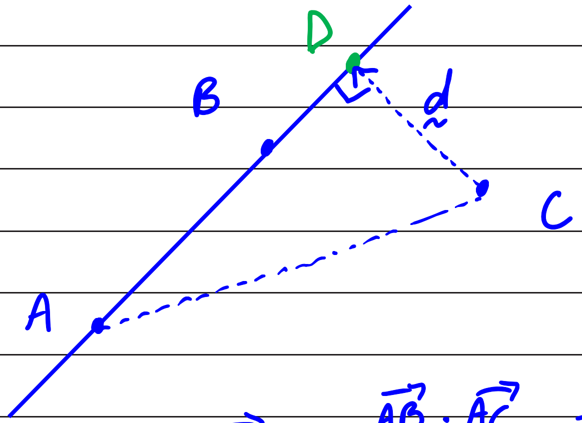
$\therefore \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + \cos^2\frac{7\pi}{12} + \cos^2\frac{11\pi}{12} = 2$

Question 14

(a) (i) $A(-2,1,7)$, $B(-1,3,5)$, $C(-4,7,3)$

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \overrightarrow{AC} = \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix}$$

(ii)



$$\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AB} \cdot \overrightarrow{AB}} \overrightarrow{AB}$$

$$= \frac{-2 + 12 + 8}{1 + 4 + 4} \overrightarrow{AB}$$

$$= \frac{18}{9} \overrightarrow{AB}$$

$$= 2\overrightarrow{AB}$$

$$= \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$$

$$\therefore \underset{\sim}{d} = \overrightarrow{AD} - \overrightarrow{AC}$$

$$= \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

and $|\underset{\sim}{d}| = \sqrt{4^2 + (-2)^2 + 0^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$ units

(b) (i) $\dfrac{1}{p^2 - x^2} = \dfrac{A}{p+x} + \dfrac{B}{p-x}$

$1 = A(p-x) + B(p+x)$

$\underline{x = p:}$ $1 = 2pB$

$B = \dfrac{1}{2p}$

$\underline{x = -p:}$ $1 = 2pA$

$A = \dfrac{1}{2p}$

(ii) $\displaystyle\int \frac{1}{p^2 - x^2}\,dx = \frac{1}{2p}\int \left(\frac{1}{p+x} + \frac{1}{p-x}\right) dx$

$= \dfrac{1}{2p}\left(\ln|p+x| - \ln|p-x|\right) + c$

$= \dfrac{1}{2p}\ln\left|\dfrac{p+x}{p-x}\right| + c$

(c) (i) $m\ddot{y} = 10m - mkv^2$

$\ddot{y} = 10 - kv^2$

Terminal velocity when $\ddot{y} = 0$:

$kV^2 = 10$

$V = \sqrt{\dfrac{10}{k}}$, taking down as positive.

$\ddot{y} = k\left(\dfrac{10}{k} - v^2\right)$

$\dfrac{dv}{dt} = k(V^2 - v^2)$

(ii) $\displaystyle\int \frac{dv}{V^2 - v^2} = \int k\,dt$

$\dfrac{1}{2V}\ln\left|\dfrac{V+v}{V-v}\right| = kt + c$ from (b)(ii)

$\ln\left|\dfrac{V+v}{V-v}\right| = 2Vkt + c_2$

$$\left|\frac{V+v}{V-v}\right| = e^{2Vkt + c_2}$$

$$\frac{V+v}{V-v} = e^{c_2} \times e^{2Vkt} \quad \text{as } v < V$$

When $t=0$, $v=0$: $\dfrac{V+0}{V-0} = e^{c_2} \times 1$

$$e^{c_2} = 1$$

$$\therefore \quad \frac{V+v}{V-v} = e^{2Vkt}$$

$$V+v = (V-v)e^{2Vkt}$$

$$v(1+e^{2Vkt}) = V(e^{2Vkt}-1)$$

$$v = V\,\frac{e^{2Vkt}-1}{e^{2Vkt}+1}$$

(iii) When $t=1$, $v=\frac{1}{2}V$:

$$\frac{1}{2}\cancel{V} = \cancel{V}\,\frac{e^{2Vk}-1}{e^{2Vk}+1}$$

$$e^{2Vk}+1 = 2e^{2Vk}-2$$

$$e^{2Vk} = 3$$

$$2Vk = \ln 3$$

but $V = \sqrt{\frac{10}{k}}$:

$$\sqrt{\frac{10}{k}} \times k = \frac{1}{2}\ln 3$$

$$\sqrt{10} \times \sqrt{k} = \frac{1}{2}\ln 3$$

$$k = \left(\frac{\ln 3}{2\sqrt{10}}\right)^2$$

$$= 0.03 \text{ (2 decimal places)}$$

Question 15

(a) $\operatorname{Im}\left(\frac{z+\bar{z}}{z-\bar{z}}\right) \geq 1$

Let $z = x+iy$: $\frac{z+\bar{z}}{z-\bar{z}} = \frac{x+iy+(x-iy)}{x+iy-(x-iy)}$

$= \frac{2x}{2iy} \times \frac{i}{i}$

$= \frac{ix}{-y}$ ✓

$\operatorname{Im}\left(-\frac{ix}{y}\right) \geq 1$

$-\frac{x}{y} \geq 1$

$\frac{x}{y} \leq -1$

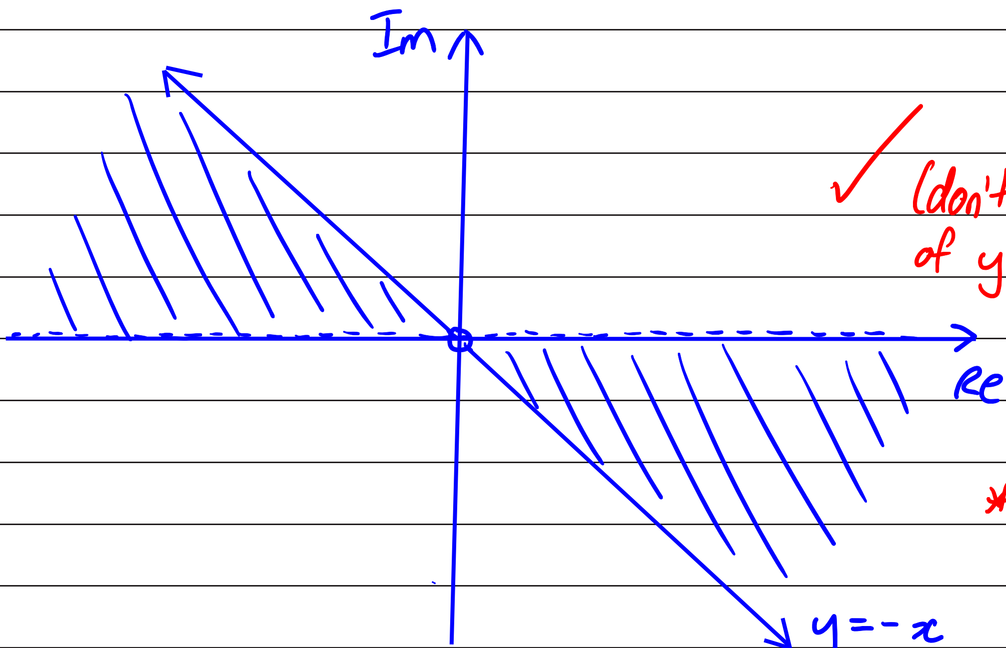
$xy \leq -y^2$, $y \neq 0$ ✓ (progress in dealing with inequality)

$y^2 + xy \leq 0$

$y(x+y) \leq 0$

So either $y > 0$ and $x+y \leq 0 \rightarrow y \leq -x$

or $y < 0$ and $x+y \geq 0 \rightarrow y \geq -x$



✓ (don't penalise no indicator of y=0 not included)

* $\frac{x}{y} \leq -1$ ↳ $y \leq -x$ + graph gets maximum 1 mark.

(b) (i) $z^7 = 1$

Let $z = \text{cis}\,\theta$

$\text{cis}\,7\theta = \text{cis}\,0$ ✓ (valid method - diagram fine)

$7\theta = 2k\pi$

$\theta = \frac{2k\pi}{7}$, $k = 0, \pm 1, \pm 2, \pm 3$

$\therefore$ the roots are ✓

1, $\text{cis}\frac{2\pi}{7}$, $\text{cis}\left(-\frac{2\pi}{7}\right)$, $\text{cis}\frac{4\pi}{7}$, $\text{cis}\left(-\frac{4\pi}{7}\right)$, $\text{cis}\frac{6\pi}{7}$, $\text{cis}\left(-\frac{6\pi}{7}\right)$.

(ii) $z^7 = 1$

$z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$ ✓

As 1 is a zero of $z - 1 = 0$,

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$ ✓

$= \left(z - \text{cis}\frac{2\pi}{7}\right)\left(z - \overline{\text{cis}\frac{2\pi}{7}}\right)\left(z - \text{cis}\frac{4\pi}{7}\right)\left(z - \overline{\text{cis}\frac{4\pi}{7}}\right)\left(z - \text{cis}\frac{6\pi}{7}\right)\left(z - \overline{\text{cis}\frac{6\pi}{7}}\right)$

$= \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$

(iii) Let $z = 1$ in identity from part (ii):

$1+1+1+1+1+1+1 = \left(2 - 2\cos\frac{2\pi}{7}\right)\left(2 - 2\cos\frac{4\pi}{7}\right)\left(2 - 2\cos\frac{6\pi}{7}\right)$ ✓

$7 = 2^3\left(1 - \cos\frac{2\pi}{7}\right)\left(1 - \cos\frac{4\pi}{7}\right)\left(1 - \cos\frac{6\pi}{7}\right)$

$7 = 2^6\left(\frac{1}{2}\left(1 - \cos\frac{2\pi}{7}\right)\right)\left(\frac{1}{2}\left(1 - \cos\frac{4\pi}{7}\right)\right)\left(\frac{1}{2}\left(1 - \cos\frac{6\pi}{7}\right)\right)$

$7 = 2^6 \sin^2\frac{\pi}{7} \times \sin^2\frac{2\pi}{7} \times \sin^2\frac{3\pi}{7}$

$\therefore \sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7} = \frac{7}{2^6}$

$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2^3}$

$= \frac{\sqrt{7}}{8}$ ✓

(c) $\left(2-\frac{1}{n}\right)^n > n, \quad n \geq 2$

(A) Prove true for $n=2$:

$$\begin{aligned} LHS &= \left(2-\tfrac{1}{2}\right)^2 \\ &= \left(\frac{3}{2}\right)^2 \\ &= \frac{9}{4} \end{aligned}$$

$$\begin{aligned} RHS &= 2 \\ &< \frac{9}{4} \end{aligned}$$

$\therefore$ true for $n=2$ ✓

(B) Assume true for some $n=k$, $k \geq 2$

i.e. $\left(2-\frac{1}{k}\right)^k > k$ (*)

Prove true for $n=k+1$:

RTP: $\left(2-\frac{1}{k+1}\right)^{k+1} > k+1$

$$\begin{aligned} LHS - RHS &= \left(2-\frac{1}{k+1}\right)^{k+1} - (k+1) \\ &> \left(2-\frac{1}{k}\right)^{k+1} - (k+1) \quad \text{as } k > 0 \\ &= \left(2-\frac{1}{k}\right)^k\left(2-\frac{1}{k}\right) - (k+1) \\ &> k\left(2-\frac{1}{k}\right) - (k+1) \quad \text{from (*)} \checkmark \\ &= 2k - 1 - k - 1 \\ &= k - 2 \\ &\geq 0 \quad \text{as } k \geq 2 \checkmark \end{aligned}$$

$\therefore$ LHS $-$ RHS > 0 and LHS $>$ RHS

From (A) and (B) above, statement holds for all integers $n \geq 2$ by mathematical induction.

(d) (i)

$$I = \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx$$

let $u = a+b-x$

$du = -dx$

$x = a,\ u = b$

$x = b,\ u = a$

$$= \int_b^a \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,-du$$

$$= \int_a^b \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,du$$

$$= \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx \quad \text{(dummy variable)}$$

$$\therefore 2I = \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx + \int_a^b \frac{f(a+b-x)\,dx}{f(a+b-x)+f(x)}$$

$$= \int_a^b \frac{f(x)+f(a+b-x)}{f(a+b-x)+f(x)}\,dx$$

$$= \int_a^b 1\,dx$$

$$= b-a$$

$$I = \frac{b-a}{2}$$

(ii)

$$\int_1^3 \frac{x^2\,dx}{\sqrt{10-x^2}+x}$$

let $u = x^2$

$du = 2x\,dx$

$x = 1,\ u = 1$

$x = 3,\ u = 9$

$$= \frac{1}{2}\int_1^3 \frac{x \cdot 2x\,dx}{\sqrt{10-x^2}+x}$$

$$= \frac{1}{2}\int_1^9 \frac{\sqrt{u}\,du}{\sqrt{10-u}+\sqrt{u}}$$

$$= \frac{1}{2} \times \frac{9-1}{2} \quad \text{from (i)}$$

$$= 2$$

Question 16

(a) (i) $\overrightarrow{z_3 z_2} = \overrightarrow{z_2 z_1}$

$$z_3 - z_2 = z_2 - z_1$$
$$2z_2 = z_3 + z_1$$
$$= z_2 z_1 + z_1$$
$$z_2(2 - z_1) = z_1$$
$$z_2 = \frac{z_1}{2 - z_1}$$

✓

(ii)(α) Let $z_1 = x + iy$

$$z_2 = \frac{x+iy}{2-x-iy} \times \frac{(2-x)+iy}{(2-x)+iy}$$
$$= \frac{x(2-x) - y^2 + i(y(2-x) + xy)}{(2-x)^2 + y^2}$$

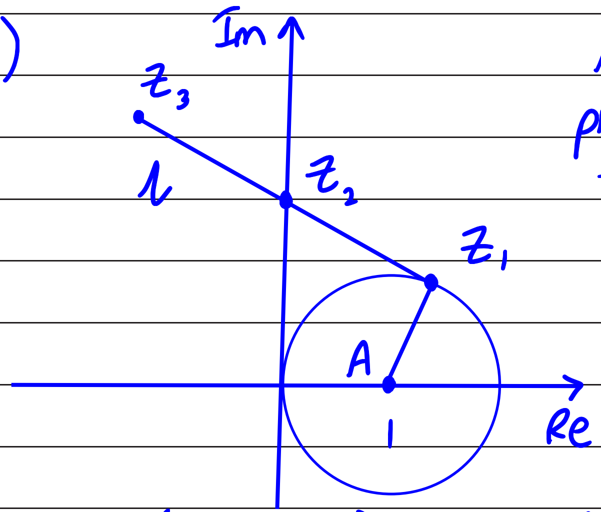
✓

As z_2 is purely imaginary, $Re(z_2) = 0$

$$\therefore \quad x(2-x) - y^2 = 0$$
$$2x - x^2 - y^2 = 0$$
$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1$$

which is a circle with centre $(1, 0)$ and radius 1. ✓

(iii)



As z_1 lies on the circle, proving that $Az_1 \perp l$ is sufficient to show that l is a tangent.

$$\text{Arg}(z_3 - z_2) = \text{Arg}(z_2 z_1 - z_2)$$
$$= \text{Arg}(z_2(z_1 - 1))$$
$$= \text{Arg}(z_2) + \text{Arg}(z_1 - 1)$$
$$= \pm\frac{\pi}{2} + \text{Arg}(z - 1)$$

✓ (progress using valid method)

As the arguments differ by $\frac{\pi}{2}$, $Az_1 \perp l$, and l is a tangent. ✓

(b) (i) For the vectors $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and
$\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$,

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$$

also, $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \times \cos\theta$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$-|\underset{\sim}{a}|\times|\underset{\sim}{b}| \le \underset{\sim}{a}\cdot\underset{\sim}{b} \le |\underset{\sim}{a}|\times|\underset{\sim}{b}|, \text{ as } -1\le\cos\theta\le 1$$

$$\therefore |\underset{\sim}{a}\cdot\underset{\sim}{b}| \le |\underset{\sim}{a}|\times|\underset{\sim}{b}|$$

$$(\underset{\sim}{a}\cdot\underset{\sim}{b})^2 \le |\underset{\sim}{a}|^2\times|\underset{\sim}{b}|^2, \text{ as LHS, RHS} \ge 0$$

$$\therefore (a_1b_1 + a_2b_2 + a_3c_3)^2 \le (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) \ge (a_1b_1+a_2b_2+a_3b_3)^2$$

(ii) $x^2 + 4y^2 + z^2 = x^2 + (2y)^2 + z^2$

Using part (i), let $a_1 = x,\ a_2 = 2y,\ a_3 = z,$
$b_1 = 1,\ b_2 = \frac{1}{2},\ b_3 = 1$:

$$\left(x^2+(2y)^2+z^2\right)\left(1^2+\left(\tfrac{1}{2}\right)^2+1^2\right) \ge \left(x\times1 + 2y\times\tfrac{1}{2} + z\times 1\right)^2$$

$$(x^2+4y^2+z^2)\left(\tfrac{9}{4}\right) \ge (x+y+z)^2$$

$$x^2+4y^2+z^2 \ge \frac{4}{9}\times 3^2 \quad \text{as } x+y+z=3$$

$$\therefore x^2+4y^2+z^2 \ge 4$$

(c) (i) $$\int_0^3 x f'(x) f(x)\,dx = \int_0^3 x f(x)\times\frac{d}{dx}(f(x))\,dx$$

$$= \left[x(f(x))^2\right]_0^3 - \int_0^3 f(x)\left(x f'(x) + f(x)\right)dx$$

$$= 3(f(3))^2 - 0 - \int_0^3\left(x f'(x) f(x) + (f(x))^2\right)dx$$

$$= 3\times5^2 - \int_0^3 x f'(x) f(x)\,dx - \int_0^3 (f(x))^2\,dx$$

$$\therefore 2\int_0^3 x f'(x) f(x)\,dx = 75 - 81$$

$$= -6$$

$$\therefore \int_0^3 x f'(x) f(x)\,dx = -3$$

or// $\int_0^3 x f'(x) f(x)\, dx$ Let $u = x$ $v' = f'(x)f(x)$

$u' = 1$ $v = \frac{1}{2}(f(x))^2$

$$= \left[x \times \frac{1}{2}(f(x))^2 \right]_0^3 - \frac{1}{2}\int_0^3 (f(x))^2\, dx$$

$$= \frac{3}{2} \times (f(3))^2 - 0 - \frac{1}{2} \times 81$$

$$= \frac{3 \times 5^2 - 81}{2}$$

$$= -3$$

(ii) $\left(f'(x)f(x) + kx\right)^2 = \left(f'(x)f(x)\right)^2 + 2k\left(xf'(x)f(x)\right) + k^2x^2$

$\therefore \int_0^3 \left(f'(x)f(x) + kx\right)^2 dx$

$$= \int_0^3 (f'(x)f(x))^2\, dx + 2k\int_0^3 x f'(x) f(x)\, dx + k^2\int_0^3 x^2\, dx$$

$$= \underset{\text{(given)}}{1} + \underset{\text{(from (i))}}{2k \times -3} + k^2\left[\frac{1}{3}x^3\right]_0^3$$

$$= 1 - 6k + k^2\left(\frac{1}{3} \times 3^3 - 0\right)$$

$$= 1 - 6k + 9k^2$$

$$\therefore \int_0^3 \left(f'(x) f(x) + kx\right)^2 dx = (1 - 3k)^2$$

Let $k = \frac{1}{3}$:

$$\int_0^3 \left(f'(x)f(x) + \frac{1}{3}x\right)^2 dx = 0$$

As $\left(f'(x)f(x) + \frac{1}{3}x\right)^2 \geq 0$, the only way the integral from 0 to 3 can be 0 is if

$$f'(x)f(x) + \frac{1}{3}x = 0$$

$$f'(x)f(x) = -\frac{1}{3}x$$

$$\int f'(x)f(x)\,dx = \int -\frac{1}{3}x\,dx$$

$$\frac{[f(x)]^2}{2} = -\frac{1}{3}\times\frac{x^2}{2} + c$$

$f(3)=5$:

$$\frac{5^2}{2} = -\frac{1}{\cancel{3}}\times\frac{3^{\cancel{2}}}{2} + c$$

$$c = \frac{5^2}{2} + \frac{3}{2}$$

$$= 14$$

$$\therefore \frac{[f(x)]^2}{2} = -\frac{1}{3}\times\frac{x^2}{2} + 14$$

$$[f(x)]^2 = 28 - \frac{1}{3}x^2$$

$$f(x) = \sqrt{28-\frac{1}{3}x^2} \quad \text{as } f(x)>0$$

$$\therefore f(0) = \sqrt{28}$$

$$= 2\sqrt{7}$$

✓